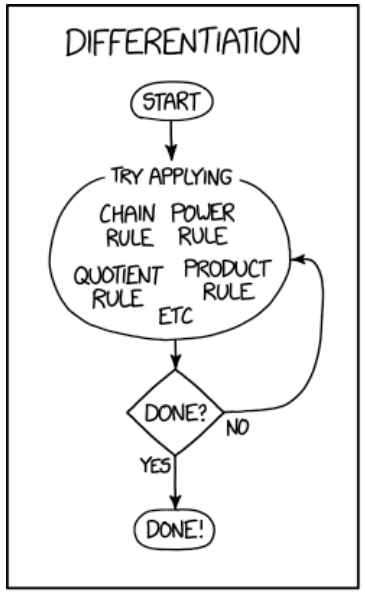
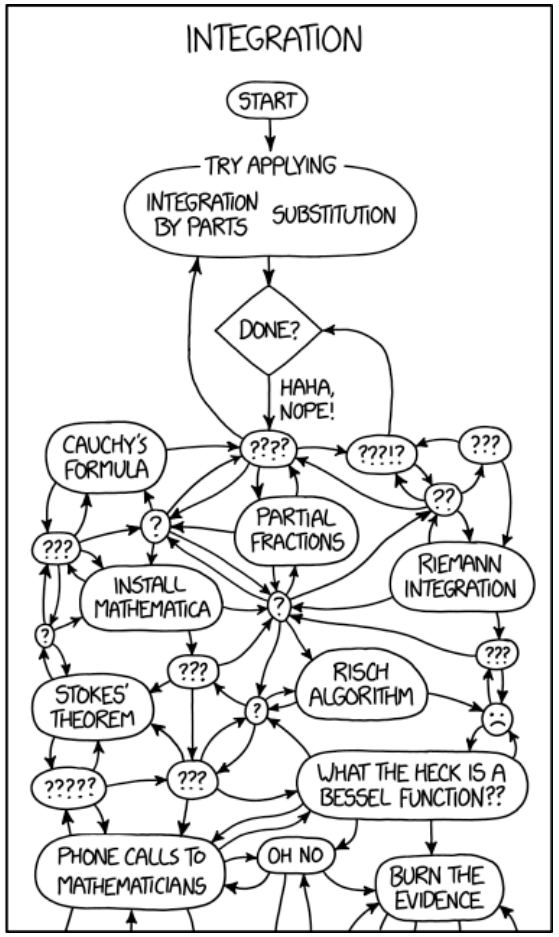
BME 2315:
Ordinary Differential Equations 1





3/27	ODEs	
4/1	ODEs	**PS 8 - regression**
4/3	ODEs	
4/8	Machine learning - intro	**PS 9 - odes**
4/10	Machine learning - intro	
4/15	Machine learning - deep learning	**PS 10 - machine learning**
4/17	Machine learning - LLMs	
4/22	MATLAB	**PS11- deep learning**
4/24	MATLAB	
4/29	Review	**PS 12- MATLAB**
5/2	**Final Exam (cumulative) 2-5pm in MR5 1041**	

Sarah Groves

UNIVERSITY of VIRGINIA

Adapted from Jeff Saucerman

Outline of next three classes

- A little bit of math/definitions
- Engineering examples
- Numerical methods to solve ODEs: Euler's method
  - Practice
  - Error estimation
  - Coding
- Midpoint method
  - Practice
- Runge Kutta 4^{th} order
  - Practice and coding

How would you solve this?

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + C$$

Integration constant → field of solutions

FIGURE PT7.3
Plots of (*a*) y versus x and (*b*) dy/dx versus x for the function $y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$.

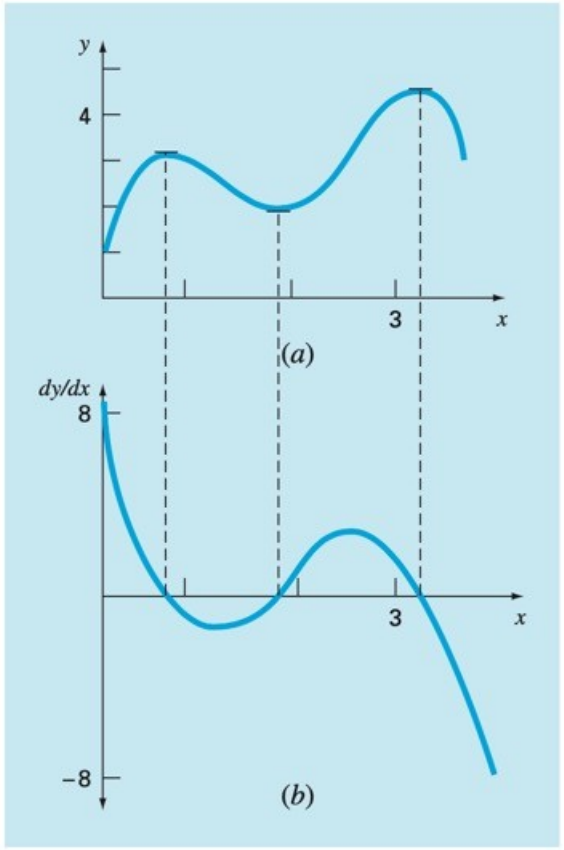
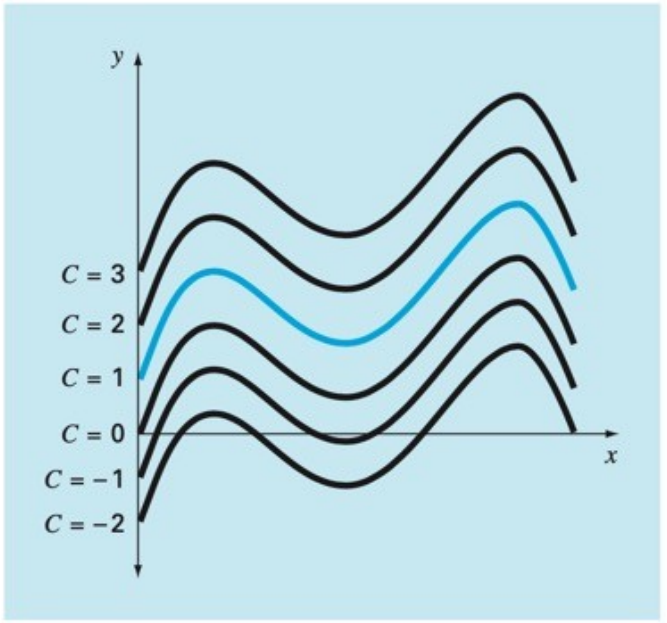
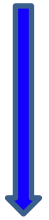


FIGURE PT7.4
Six possible solutions for the integral of $-2x^3 + 12x^2 - 20x + 8.5$. Each conforms to a different value of the constant of integration C.



What about if dy/dx also depends on y?

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$



$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5 + 2y$$

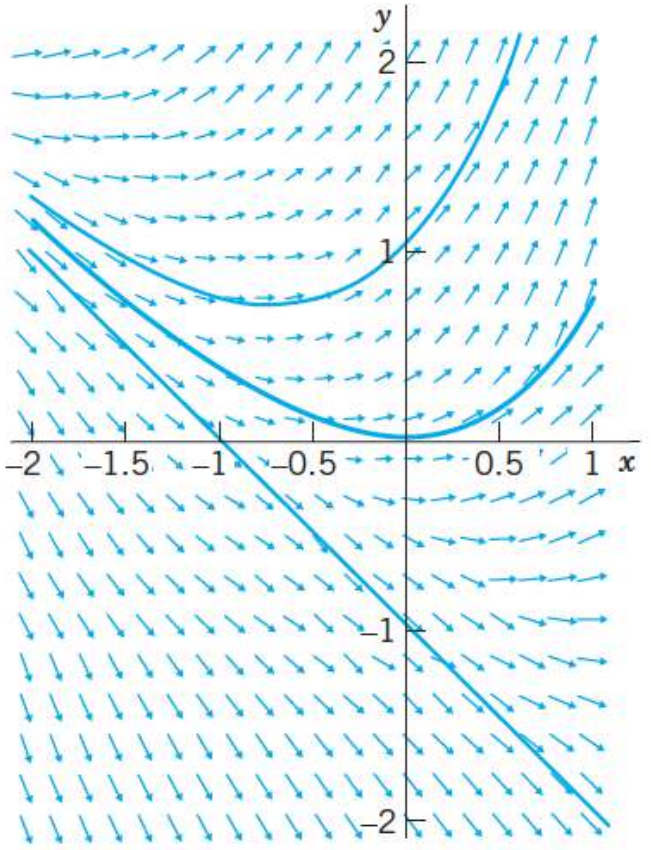
Ordinary differential equations (ODEs)

$$y'(x) = f(x, y(x))$$

independent variable

dependent variable

A solution is any curve $y(x)$ that satisfies equation above.



Exampl

e: $y'(x) = y(x) + x$

x_0

$y_0 = y(x_0)$

CHANGE IN DEPENDENT VARIABLE WITH RESPECT TO INDEPENDENT VARIABLE

Ordinary Differential Eq.

1 independent variable

Partial Differential Eq.

2+ independent variables

$$\frac{\partial C(x,y)}{\partial x} \qquad \frac{\partial C(x,y)}{\partial y}$$

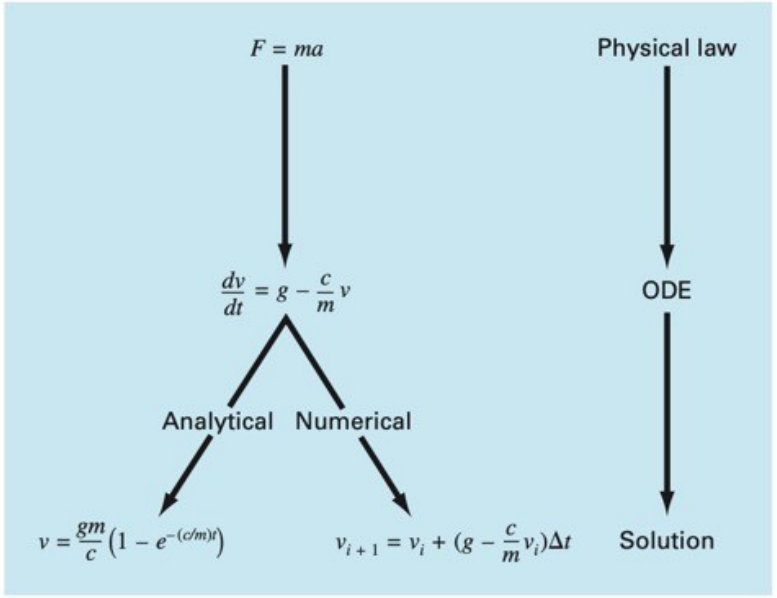
$$\frac{\partial C(t,x)}{\partial t} = D\frac{\partial^2 C(t,x)}{\partial x^2}$$

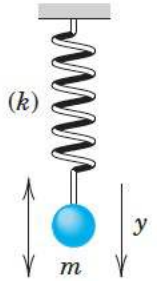
$$m\frac{d^2p(t)}{dt^2} + c\frac{dp(t)}{dt} + kp = 0 \quad \begin{cases} y = \dfrac{dp(t)}{dt} \\ \dfrac{dy}{dt} = \dfrac{d^2p(t)}{dt^2} \end{cases} \quad \begin{cases} \dfrac{dy}{dt} = \dfrac{-cy - kp}{m} \\ \dfrac{dp(t)}{dt} = y \end{cases}$$

Identify *independent* and *dependent* variables

Why?

modelin
g

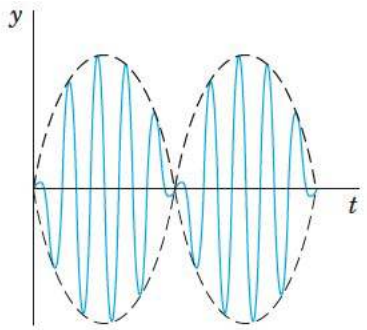




Displacement y

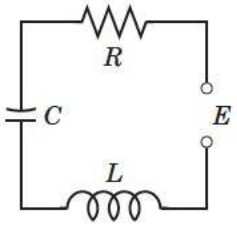
Vibrating mass on a spring

$my'' + ky = 0$



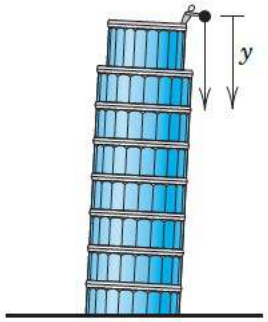
Beats of a vibrating system

$y'' + \omega_0^2 y = \cos\ \omega t, \quad \omega_0 \approx \omega$



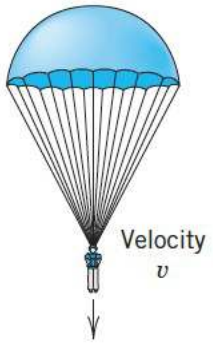
Current I in an RLC circuit

$LI'' + RI' + \frac{1}{C}I = E'$



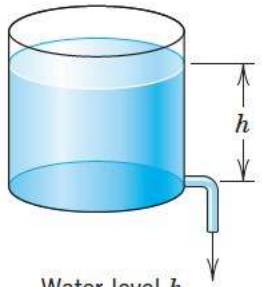
Falling stone

$y'' = g = const.$



Parachutist

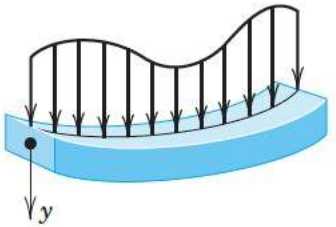
$mv' = mg - bv^2$



Water level h

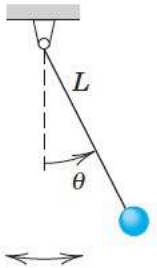
Outflowing water

$h' = -k\sqrt{h}$



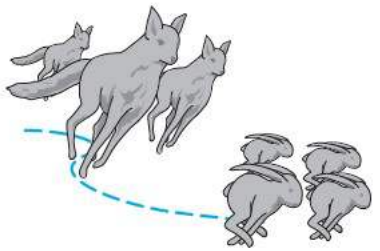
Deformation of a beam

$EIy^{iv} = f(x)$



Pendulum

$L\theta'' + g \sin\theta = 0$

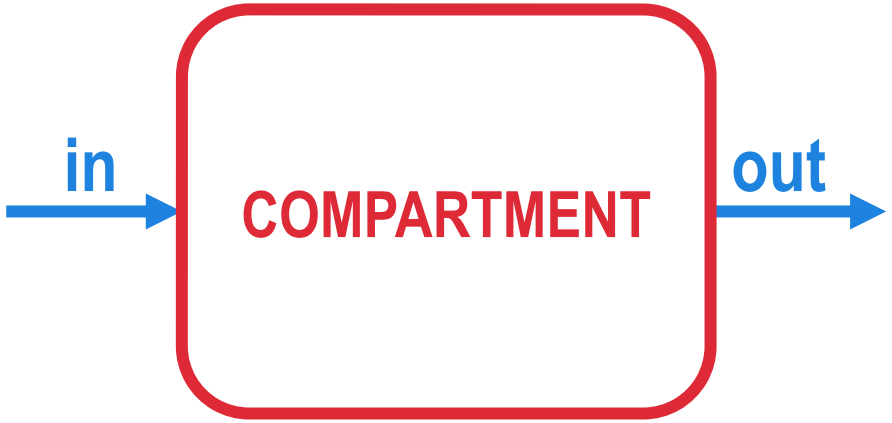


Lotka–Volterra predator–prey model

$y_1' = ay_1 - by_1y_2$

$y_2' = ky_1y_2 - ly_2$

Example: compartmental models



BME Applications

Reaction Kinetics: drugs, enzyme-ligand

Mass/Energy Transfer: blood flow, epidemiology

How do we solve ODEs?

1) Guessing

E.g.: exponential growth/decay

$$y(t) = y_0 e^{kt}$$

$$y'(t) = k y_0 e^{kt} = k y(t)$$

$$y' = ky$$

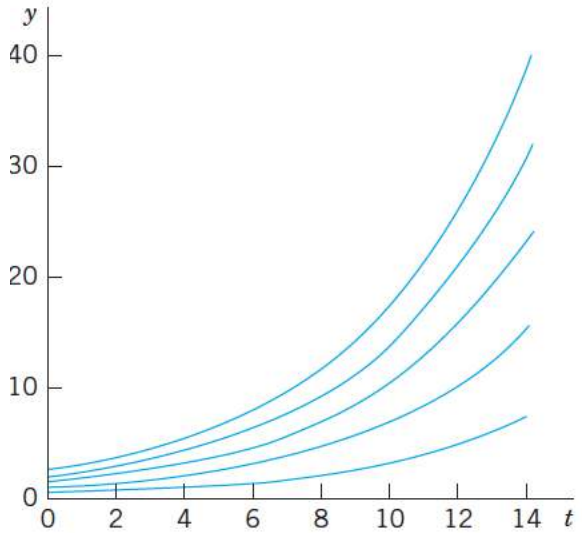


Fig. 4A. Solutions of $y' = 0.2y$ in Example 3 **(exponential growth)**

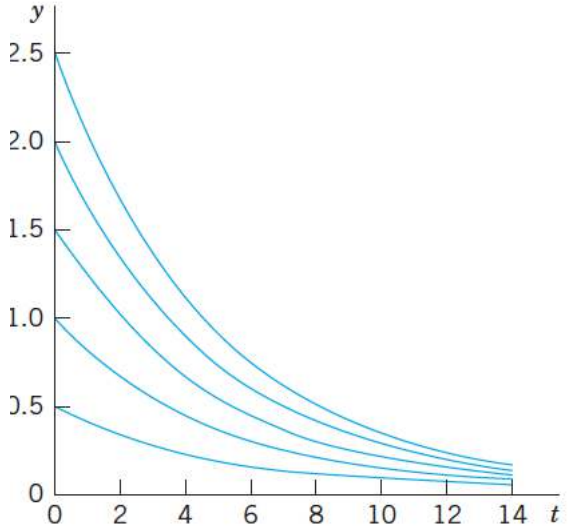


Fig. 4B. Solutions of $y' = -0.2y$ in Example 3 **(exponential decay)**

How do we solve ODEs?

2) Separation of variables

$$g(y)\,y' = f(x) \longrightarrow \int g(y)\,y'\,dx = \int f(x)\,dx + c \longrightarrow \int g(y)\,dy = \int f(x)\,dx + c.$$

exampl e

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1+y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c).$$

How do we solve ODEs?

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

The initial condition at $x = 0$ is $y = 1$.

3. Numerical methods

ODE Form: $\frac{dy}{dt} = f(t, y)$

General Form:

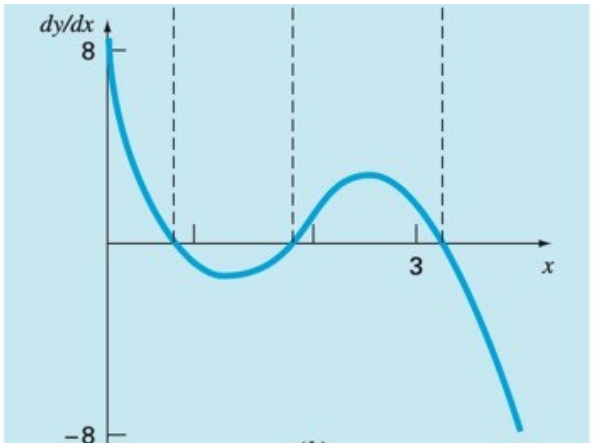
$$y_{i+1} = y_i + \phi h$$

$$\phi = f(t_i, y_i)$$

Euler's method

Where have you seen this equation before?

$y_{i+1}=y_i+y_i'*h$



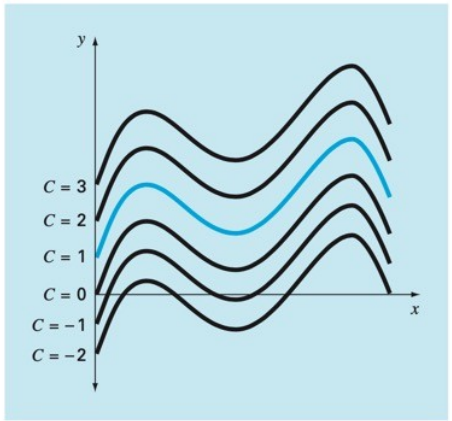
We know:

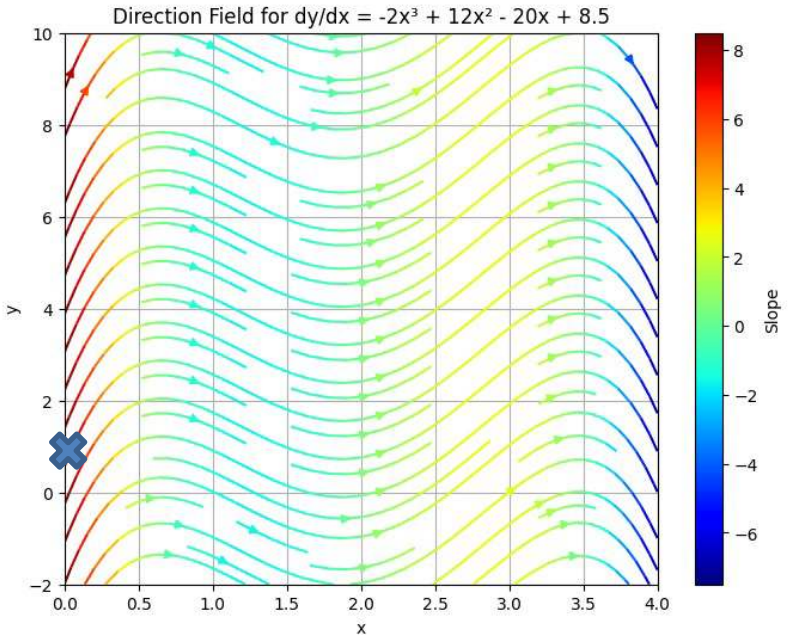
- $y'=f(x)$ at any x, including y_i' and y_{i+1}'
- $y(x_i) = y_i$

We don't know:

- $y(x_{i+1}) = y_{i+1}$

True solution





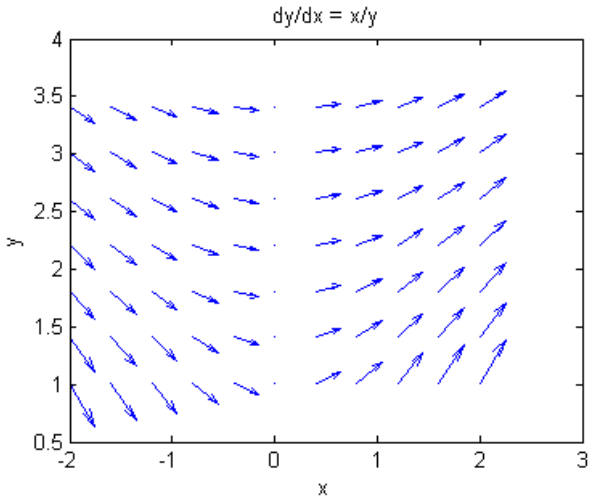
Euler's Method Example

$$\frac{dy}{dx} = \frac{x}{y}$$

Solve y(x) using Euler's method.

I.C. $y(x = -2) = 3$

$h = 1$



iteration				
1				
2				
3				

Euler's Method for systems of ODEs

Single ODE: $y_{i+1} = y_i + f(x_i, y_i)h$

System of ODEs: $Y_{i+1} = Y_i + F(x_i, Y_i)h$

$$Y_i = \begin{bmatrix} y_1 \\ y_2 \\ \dots \end{bmatrix}, \mathrm{F(x, Y)} = \begin{bmatrix} f_1(x, Y) \\ f_2(x, Y) \\ \dots \end{bmatrix}$$

System of ODEs example

$$\frac{dy_1}{dt} = y_1 y_2 = f_1$$

$$\frac{dy_2}{dt} = 3t - y_1 = f_2$$

$$Y_{i+1} = Y_i + F(x_i, Y_i)h$$

I.C.s

$y_1(t = 0) = 1$

$y_2(t = 0) = 2$

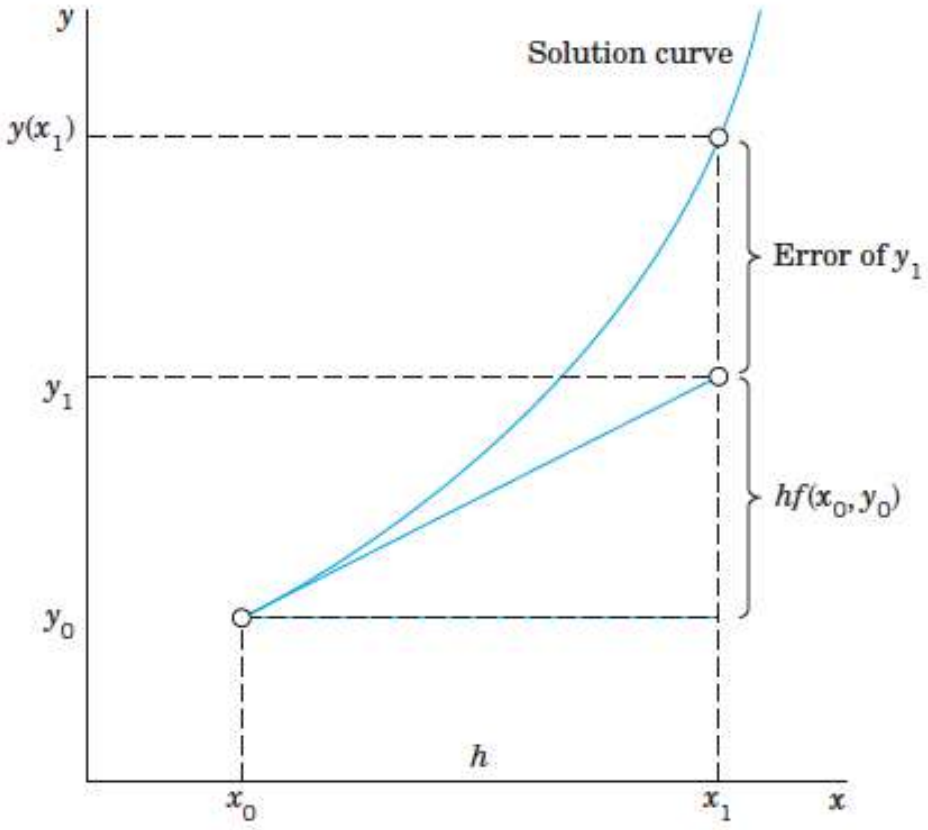
$h = 0.5$

iteration	iteration	t_i	$\vec{Y}_i$	$\vec{F}(t_i, \vec{Y}_i)$	$\vec{Y}_{i+1}$	iteration	t_i	$\vec{Y}_i$	$\vec{F}(t_i, \vec{Y}_i)$	$\vec{Y}_{i+1}$	iteration	t_i	$\vec{Y}_i$	$\vec{F}(t_i, \vec{Y}_i)$	$\vec{Y}_{i+1}$
	1					1					1					
	2					2					2					
1																
2																

Euler’s method practice!

Taylor Series error estimate for Euler's Method

Local error: error at t_{i+1} from t_i



Global error: error at time t after how many steps it takes to get there from t0

Python code for Euler's method

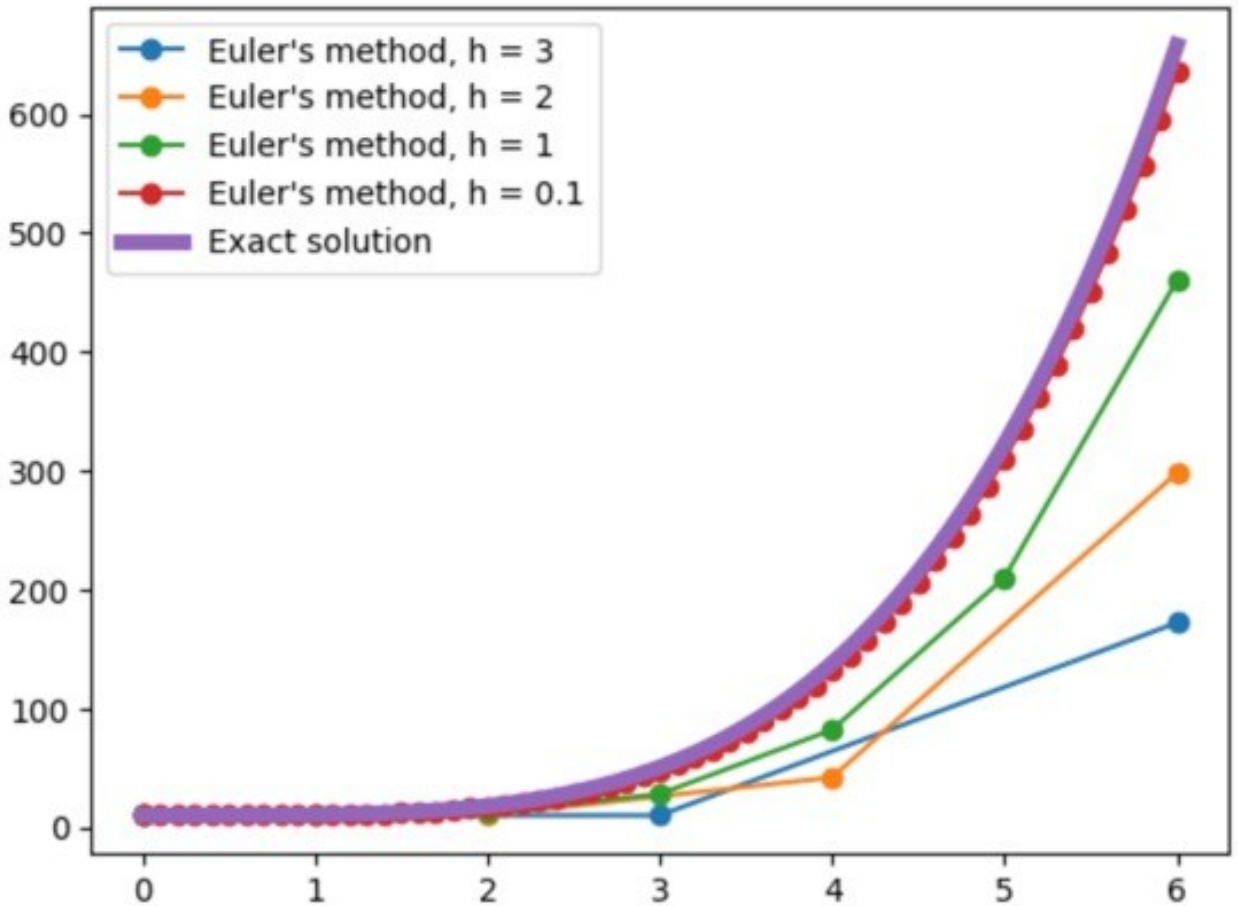
```
# eulers.py
import numpy as np


def eulers(ODEfunc,tspan,y0,h):

        return t,y
```

$$\frac{dy}{dt} = 2t^3$$

Why are the predictions for y_{i+1} underestimating the true value?



Key Points

- ODEs can be used to model many different biological systems
- Euler's Method
  - Approximate solution using slope at current location
  - Error depends on step size
  - Extend directly to a system of ODEs